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Output Tracking with Nonhyperbolic and Near Nonhyperbolic Internal Dynamics: Helicopter Hover Control

Santosh Devasia*
University of Utah, Salt Lake City, Utah 84112

A technique to achieve output tracking for nonminimum phase linear systems with nonhyperbolic and near nonhyperbolic internal dynamics is presented. This approach integrates stable inversion techniques, which achieve exact tracking, with approximation techniques, which modify the internal dynamics, to achieve desirable performance. Such modification of the internal dynamics is used 1) to remove nonhyperbolicity, which is an obstruction to applying stable inversion techniques, and 2) to reduce large preactuation times needed to apply stable inversion for near nonhyperbolic cases. The method is applied to an example helicopter hover control problem with near nonhyperbolic internal dynamics for illustrating the tradeoff between exact tracking and reduction of preactuation time.

I. Introduction

PRECISION output tracking controllers are needed to meet increasingly stringent performance requirements in applications such as flexible structures, aircraft and air traffic control, robotics, and manufacturing systems. Although perfect tracking of minimum phase systems is relatively easy to achieve, output tracking of nonminimum phase systems tends to be more challenging due to fundamental limitations on transient tracking performance.[1] This poor transient performance has been mitigated by using preactuation in the stable inversion-based approaches for nonminimum phase systems.[2–5] However, the required preactuation time (during which most of the preactuation control effort is required) is large if the zeros of a linear system that lie on the open right-half of the complex plane are close to the imaginary axis. In the limiting case, with the zeros on the imaginary axis (nonhyperbolic internal dynamics), presently available inversion-based techniques fail because the preactuation time needed becomes infinite. We present a design technique for output tracking of linear nonminimum phase systems, which have nonhyperbolic and near nonhyperbolic nonminimum phase internal dynamics. This technique is then applied to an example helicopter hover control problem, and simulation results are presented.

Output tracking has a long history marked by the development of regulator theory for linear systems by Francis and Wonham[6] and the generalization to the nonlinear case by Isidori and Byrnes.[7] These approaches asymptotically track an output from a class of exosystem-generated outputs. Further, the Isidori–Byrnes regulator has been extended in Refs. 8 and 9 and computational issues have been studied in Refs. 10 and 11. Although nonlinear regulator design is computationally difficult, the linear regulator is easily designed by solving a manageable set of linear equations. A problem, however, with the regulator approach is that the exosystem states are often switched to describe the desired output; this leads to transient tracking errors after the switching instants. Such switching-caused transient errors can be avoided by using inversion-based approaches to output tracking.[4,12] Thus, it is advantageous to use inversion-based output tracking when precision tracking of a particular output trajectory is required.

Inversion, which is key to our approach, was restricted to causal inverses of minimum phase systems in the early works by Silverman[13] and Hirschorn[14] because these approaches lead to unbounded inverses in the nonminimum phase case. Di Benedetto and Lucibello[15] considered the inversion of time varying nonminimum phase systems with a choice of the system's initial conditions. Instead of choosing initial conditions, preactuation was used by noncausal stable inversion techniques developed in Refs. 2–5 and 16. Such noncausal inverses, which require preactuation, have been successfully applied to the output tracking of flexible structures[17,18] and aircraft and air traffic control.[19,20] However, the fundamental limitation of presently available inversion schemes is that they fail if the internal dynamics is nonhyperbolic. Even when the internal dynamics is hyperbolic, if the right-half-plane zeros of the system are close to the imaginary axis (the near nonhyperbolic case), then the required preactuation time tends to become unacceptably large. In summary, output tracking remains a challenge for nonminimum phase systems with nonhyperbolic or near nonhyperbolic internal dynamics.

There are several approximation-based output tracking techniques, where the central philosophy is to replace the internal dynamics with a dynamics that provides satisfactory behavior, and then to develop the controller based on the altered system.[21–23] The technique most relevant to this paper is developed by Gopalswamy and Hedrick,[23] where trajectory modifications are considered to stabilize the internal dynamics. This technique, however, requires hyperbolicity of the internal dynamics for computational purposes. The development of computational techniques for stable inversion (e.g., Ref. 16) motivates the present integration of the stable inversion scheme with approximation techniques, especially for systems with nonhyperbolic internal dynamics where the existing stable inversion techniques fail. However, instead of stabilizing the unstable internal dynamics, we only aim to modify the nonhyperbolic behavior with a small perturbation of the internal dynamics. Additionally, in nonminimum phase systems with near nonhyperbolic internal dynamics, the present approach allows a tradeoff between the precision tracking requirement and the amount of preactuation time needed to apply the stable inversion-based output tracking technique.

The approximate inversion-based technique is developed in Sec. II, and the technique is applied to a helicopter hover control example in Sec. III, where simulation results are discussed. Conclusions are in Sec. IV.

II. Stable Inversion

A. Inversion-Based Output Tracking Scheme

Here we describe how the inversion approach is used to develop output tracking controllers. Consider a linear system described by

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t), \qquad \boldsymbol{y}(t) = C\boldsymbol{x}(t) \tag{1}$$

which has the same number of inputs as outputs, $\boldsymbol{u}(t), \boldsymbol{y}(t) \in \Re^p$, and $\boldsymbol{x}(t) \in \Re^n$. Let $\boldsymbol{y}_d(\cdot)$ be the desired output trajectory to be tracked. Then in the inversion-based approach, first, we find a nominal input-state trajectory $[\boldsymbol{u}_{\text{ff}}(\cdot), \boldsymbol{x}_{\text{ref}}(\cdot)]$ that satisfies the system equations (1) and yields the desired output exactly, i.e.,

$$\left.\begin{aligned} \dot{\boldsymbol{x}}_{\text{ref}}(t) &= A\boldsymbol{x}_{\text{ref}}(t) + B\boldsymbol{u}_{\text{ff}}(t) \\ \boldsymbol{y}_d(t) &= C\boldsymbol{x}_{\text{ref}}(t) \end{aligned}\right\} \qquad \forall t \in (-\infty, \infty) \tag{2}$$

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*Assistant Professor, Department of Mechanical Engineering. Member AIAA.

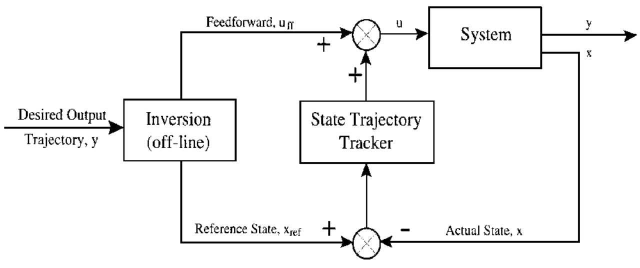

Fig. 1 Control scheme.

Second, we stabilize the exact-output yielding state trajectory $\boldsymbol{x}_{\mathrm{ref}}(\cdot)$ by using state feedback (see Fig. 1). Thus, $\boldsymbol{x}(t) \to \boldsymbol{x}_{\mathrm{ref}}(t)$ and $\boldsymbol{y}(t) \to \boldsymbol{y}_d(t)$ as $t \to \infty$, and output tracking is achieved. It is noted that in this output tracking scheme, the reference state trajectory $\boldsymbol{x}_{\mathrm{ref}}(\cdot)$ and the feedforward input $\boldsymbol{u}_{\mathrm{ff}}(\cdot)$ are computed off-line. Whereas stabilization of the reference state trajectory can be easily achieved through standard techniques[24] such as state feedback of the form $K[\boldsymbol{x}(t) - \boldsymbol{x}_{\mathrm{ref}}(t)]$, the main challenge is to find the inverse input-state trajectory $[\boldsymbol{u}_{\mathrm{ff}}(\cdot), \boldsymbol{x}_{\mathrm{ref}}(\cdot)]$, especially for systems with nonminimum phase dynamics. This paper addresses the off-line computation of the inverse input-state trajectory for a given desired trajectory, $\boldsymbol{y}_d(\cdot)$.

B. Internal Dynamics

In this subsection, it is shown that finding the inverse input-state trajectory is equivalent to finding bounded solutions to the system's internal dynamics. Let the linear system (1) have a well-defined vector relative degree, $\boldsymbol{r} := [r_1, r_2, \ldots, r_p]$. Then the output's derivatives are given as[25]

$$\frac{\mathrm{d}^{r_k} y_k}{\mathrm{d}t^{r_k}} = C_k A^{r_k} \boldsymbol{x} + C_k A^{r_k - 1} B \boldsymbol{u} \tag{3}$$

where C_k is the kth row of C and $1 \le k \le p$. In vector notation, let Eq. (3) be rewritten as

$$\boldsymbol{y}^{(r)}(t) = A_x \boldsymbol{x}(t) + B_y \boldsymbol{u}(t) \tag{4}$$

where

$$\boldsymbol{y}^{(r)} := \begin{bmatrix} \frac{\mathrm{d}^{r_1} y_1}{\mathrm{d}t^{r_1}} & \frac{\mathrm{d}^{r_2} y_2}{\mathrm{d}t^{r_2}} & \cdots & \frac{\mathrm{d}^{r_p} y_p}{\mathrm{d}t^{r_p}} \end{bmatrix}^T$$

$$A_x := \begin{bmatrix} C_1 A^{r_1} \\ C_2 A^{r_2} \\ \vdots \\ C_p A^{r_p} \end{bmatrix} \qquad B_y := \begin{bmatrix} C_1 A^{r_1 - 1} B \\ C_2 A^{r_2 - 1} B \\ \vdots \\ C_p A^{r_p - 1} B \end{bmatrix}$$

and B_y is invertible because of the well-defined relative degree assumption. Equation (4) motivates the choice of the control law of the form

$$\boldsymbol{u}_{\mathrm{ff}}(t) = B_y^{-1}\left[\boldsymbol{y}_d^{(r)}(t) - A_x \boldsymbol{x}(t)\right] \tag{5}$$

for all $t \in (-\infty, \infty)$. Substituting this control law in Eq. (4), it is seen that exact tracking is maintained, i.e.,

$$\boldsymbol{y}^{(r)}(t) = \boldsymbol{y}_d^{(r)}(t)$$

To study the effect of this control law, consider a change of coordinates T such that[25]

$$\begin{bmatrix} \boldsymbol{\zeta}(t) \\ \boldsymbol{\eta}(t) \end{bmatrix} = T\boldsymbol{x}(t)$$

where $\boldsymbol{\zeta}(t)$ consists of the output and its time derivatives

$$\boldsymbol{\zeta}(t) := \left[y_1, \dot{y}_1, \ldots, \frac{\mathrm{d}^{r_1 - 1}}{\mathrm{d}t^{r_1 - 1}} y_1, y_2, \dot{y}_2, \ldots, \frac{\mathrm{d}^{r_2 - 1}}{\mathrm{d}t^{r_2 - 1}} y_2, \ldots, y_p, \dot{y}_p, \ldots, \frac{\mathrm{d}^{r_p - 1}}{\mathrm{d}t^{r_p - 1}} y_p \right]'$$

The system equation (1) can then be rewritten in the new coordinates as

$$\dot{\boldsymbol{\zeta}}(t) = \hat{A}_1 \boldsymbol{\zeta} + \hat{A}_2 \boldsymbol{\eta} + \hat{B}_1 \boldsymbol{u} \tag{6}$$

$$\dot{\boldsymbol{\eta}}(t) = \hat{A}_3 \boldsymbol{\zeta} + \hat{A}_4 \boldsymbol{\eta} + \hat{B}_2 \boldsymbol{u} \tag{7}$$

where

$$\hat{A} := T^{-1} A T := \begin{bmatrix} \hat{A}_1 & \hat{A}_2 \\ \hat{A}_3 & \hat{A}_4 \end{bmatrix} \quad \text{and} \quad \hat{B} := \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} = T^{-1} B$$

In the new coordinates, the control law for maintaining exact tracking [Eq. (5)] can be written as

$$\boldsymbol{u}(t) = B_y^{-1}\left[\boldsymbol{y}_d^{(r)}(t) - A_\zeta \boldsymbol{\zeta}_d(t) - A_\eta \boldsymbol{\eta}(t)\right] \tag{8}$$

where

$$\begin{bmatrix} A_\zeta \\ A_\eta \end{bmatrix} := A_x T^{-1}$$

Note that the desired $\boldsymbol{\zeta}(\cdot)$ is known when the desired output trajectory $\boldsymbol{y}_d(\cdot)$ and the output's time derivatives are specified. This desired $\boldsymbol{\zeta}(\cdot)$ is defined as $\boldsymbol{\zeta}_d(\cdot)$. Inasmuch as the control law was chosen such that exact tracking is maintained, $\boldsymbol{y}^{(r)}(t) = \boldsymbol{y}_d^{(r)}(t)$, we also have $\dot{\boldsymbol{\zeta}}(t) = \dot{\boldsymbol{\zeta}}_d(t)$, and Eqs. (6) and (7) become

$$\dot{\boldsymbol{\zeta}}(t) = \dot{\boldsymbol{\zeta}}_d(t) \tag{9}$$

$$\dot{\boldsymbol{\eta}}(t) = \hat{A}_3 \boldsymbol{\zeta} + \hat{A}_4 \boldsymbol{\eta} + \hat{B}_2 B_y^{-1}\left[\boldsymbol{y}_d^{(r)}(t) - A_\zeta \boldsymbol{\zeta}_d(t) - A_\eta \boldsymbol{\eta}(t)\right] \tag{10}$$

This is the inverse system, and in particular, Eq. (10) is the internal dynamics. Solving the internal dynamics is key to finding the inverse input-state trajectories. If a bounded solution $\boldsymbol{\eta}_d(\cdot)$ to the internal dynamics (10) can be found, then the feedforward input can be found through Eq. (8) as

$$\boldsymbol{u}_{\mathrm{ff}}(t) = B_y^{-1}\left[\boldsymbol{y}_d^{(r)}(t) - A_\zeta \boldsymbol{\zeta}_d(t) - A_\eta \boldsymbol{\eta}_d(t)\right] \tag{11}$$

and the reference trajectory can be found as

$$\boldsymbol{x}_{\mathrm{ref}}(t) = T^{-1} \begin{bmatrix} \boldsymbol{\zeta}_d(t) \\ \boldsymbol{\eta}_d(t) \end{bmatrix}$$

Thus, a bounded solution to the internal dynamics (10) is required to find the inverse and to apply the output tracking scheme shown in Fig. 1.

C. Modified Internal Dynamics

Standard inversion schemes[13,14] that integrate (forward in time) the internal dynamics (10) lead to unbounded solutions because the internal dynamics is unstable for nonminimum phase systems. Noncausal inversion (e.g., Ref. 4) leads to a bounded but noncausal solution to the internal dynamics. Such stable inversion techniques are, however, not applicable to systems with nonhyperbolic internal dynamics. In this subsection a compromise between stable inversion and approximation-based inversion schemes is proposed. The key is to modify the internal dynamics by giving up exact output tracking, enough to remove the nonhyperbolicity, and then to apply stable inversion. Note that the system zeros are not being modified by output feedback (which is impossible); rather, the inverse system is perturbed to a nearby system, which has better behaved internal dynamics, for stable inversion. The difference between the proposed technique and other approximation techniques is that the internal dynamics is perturbed only to remove the nonhyperbolicity and not to stabilize the entire internal dynamics.

To change the nonhyperbolicity of the internal dynamics, an extra term, $\boldsymbol{v}(t)$, is added to the control law (8) as follows:

$$\boldsymbol{u}_{\mathrm{ff}}(t) = B_y^{-1}\left[\boldsymbol{y}_d^{(r)}(t) - A_\zeta \boldsymbol{\zeta}(t) - A_\eta \boldsymbol{\eta}(t) + \boldsymbol{v}(t)\right] \tag{12}$$

With this modified control law, the inverse system [Eqs. (9) and (10)] becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} = \hat{S}\begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} + G_y Y_d(t) + G_v v(t) \tag{13}$$

where $e_\zeta(t) := \zeta(t) - \zeta_d(t)$ is the error in the output and the output's derivatives,

$$\hat{S} = \begin{bmatrix} \hat{A}_1^* & 0 \\ (\hat{A}_3 - \hat{B}_2 B_y^{-1} A_\zeta) & (\hat{A}_4 - \hat{B}_2 B_y^{-1} A_\eta) \end{bmatrix}, \quad G_v = \begin{bmatrix} \hat{B}_1 B_y^{-1} \\ \hat{B}_2 B_y^{-1} \end{bmatrix}$$

$$G_y = \begin{bmatrix} 0 & 0 \\ (\hat{A}_3 - \hat{B}_2 B_y^{-1} A_\zeta) & \hat{B}_2 B_y^{-1} \end{bmatrix}, \quad Y_d(t) = \begin{bmatrix} \zeta_d(t) \\ y_d^{(r)}(t) \end{bmatrix}$$

$$\hat{A}_1^* := \mathrm{diag}[A_1, A_2, \ldots, A_p]$$

where each A_i is an $r_i \times r_i$ zero-matrix with ones on the elements above the main diagonal (for all $1 \le i \le p$).

Assuming that the original system (A, B) is controllable, we also have $(\hat{S}, G_v)$ controllable, and hence there exists a feedback of the form

$$v(t) = F\begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} \tag{14}$$

such that the modified inverse system (13) is hyperbolic; i.e., all poles on the imaginary axis are moved. Note that this change to an hyperbolic system can be achieved through arbitrarily small F because nonhyperbolicity is not a structurally stable property. The hyperbolic system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} = (\hat{S} + G_v F)\begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} + G_y Y_d(t)$$

$$:= S\begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} + G_y Y_d(t) \tag{15}$$

is the modified inverse system. This modification of the internal dynamics can also be used to move unstable poles of the inverse system that may be close to the imaginary axis for reducing the required preactuation time. Next, stable inversion of the modified inverse system is carried out.[4]

D. Computation of the Inverse

We begin by decoupling the modified inverse system (15) into stable (z_s) and unstable (z_u) subsystems. Because the modified inverse system is hyperbolic, there exits a decoupling transformation U such that the modified inverse system (15) can be written as

$$\dot{z}_s(t) = S_s z_s(t) + G_s Y_d(t), \qquad \dot{z}_u(t) = S_u z_u(t) + G_u Y_d(t) \tag{16}$$

where

$$z(t) := \begin{bmatrix} z_s(t) \\ z_u(t) \end{bmatrix} = U\begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} \tag{17}$$

To find bounded solutions to the unstable inverse systems, the boundary conditions that $z_s(-\infty) = 0$ and $z_u(\infty) = 0$ are applied to Eq. (16). This leads to unique bounded solutions to the modified inverse system by flowing the stable subsystem forward in time and flowing the unstable system backward in time as

$$z_{s,d}(t) = \int_{-\infty}^{t} e^{S_s(t-\upsilon)} G_u Y_d(\upsilon)\,\mathrm{d}\upsilon \qquad \forall t \in (-\infty, \infty)$$

$$z_{u,d}(t) = -\int_{t}^{\infty} e^{S_u(t-\upsilon)} G_u Y_d(\upsilon)\,\mathrm{d}\upsilon \qquad \forall t \in (-\infty, \infty) \tag{18}$$

This completes the technique. To summarize, the bounded solution (18) is used to find the reference state trajectory as $x_{\mathrm{ref}}(t) = T^{-1}U^{-1}z_d(t)$ and to find the feedforward input $u_{\mathrm{ff}}(\cdot)$ from Eq. (12). This inverse, $[u_{\mathrm{ff}}(\cdot), x_{\mathrm{ref}}(\cdot)]$, is then used in the control scheme shown in Fig. 1 to obtain output tracking.

E. Preactuation Time and Unstable Poles of the Inverse System

The connection between the amount of preactuation time required to apply the inversion-based feedforward input and the unstable poles of the modified inverse is established in the following Lemma.

Lemma:

1) Let the support of $Y_d(\cdot)$ be contained in $[t_0, \infty)$ for some t_0.

2) Let all of the unstable poles of internal dynamics represented by the eigenvalues of S_u lie to the right, in the complex plane, of the line $\mathrm{Re}(s) = \alpha$ for some positive α.

3) Let $\|G_u Y_d(\cdot)\|_\infty < \beta$.

Then there exists M such that $\|u_{\mathrm{ff}}(t)\|_\infty < M e^{\alpha(t-t_0)}$ for all time before the start of the maneuver, $t < t_0$.

Proof: From condition 2 of the Lemma, there exists a positive constant M_{S_u} such that

$$\left\| e^{S_u(t-\upsilon)} \right\|_\infty < M_{S_u} e^{\alpha(t-\upsilon)} \qquad \forall t < \upsilon \tag{19}$$

Then for all $t < t_0$,

$$\|u_{\mathrm{ff}}(t)\|_\infty = \left\| B_y^{-1}\left[y_d^{(r)}(t) - A_\zeta \zeta(t) - A_\eta \eta(t) + v(t) \right] \right\|_\infty$$

from Eq. (12)

$$= \left\| B_y^{-1}[-A_\zeta \zeta - A_\eta \eta(t) + v(t)] \right\|_\infty$$

from condition 1 of Lemma

$$= \left\| \left(\begin{bmatrix} 0 & 0 \\ -B_y^{-1} A_\zeta & -B_y^{-1} A_\eta \end{bmatrix} + F \right) \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} \right\|_\infty$$

from Eq. (14) and condition 1 of Lemma

$$:= \left\| A_U \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} \right\|_\infty$$

$$\le \|A_U\|_\infty \left\| \begin{bmatrix} e_\zeta(t) \\ \eta(t) \end{bmatrix} \right\|_\infty$$

$$\le \|A_U\|_\infty \|U^{-1}\|_\infty \left\| \begin{bmatrix} z_s(t) \\ z_u(t) \end{bmatrix} \right\|_\infty \quad \text{from Eq. (17)}$$

$$= \|A_U\|_\infty \|U^{-1}\|_\infty \|z_u(t)\|_\infty$$

because $z_s(t) = \mathbf{0}$ for all $t < t_0$

$$= \|A_U\|_\infty \|U^{-1}\|_\infty \left\| \int_t^\infty e^{S_u(t-\upsilon)} G_u Y_d(\upsilon)\,\mathrm{d}\upsilon \right\|_\infty$$

from Eq. (18)

$$= \|A_U\|_\infty \|U^{-1}\|_\infty \left\| \int_{t_0}^\infty e^{S_u(t-\upsilon)} G_u Y_d(\upsilon)\,\mathrm{d}\upsilon \right\|_\infty$$

from condition 1 of Lemma

$$< \beta M_{S_u} \|A_U\|_\infty \|U^{-1}\|_\infty \int_{t_0}^\infty e^{\alpha(t-\upsilon)}\,\mathrm{d}\upsilon$$

from Eq. (19) and condition 3 of the Lemma. Integrating the preceding expression, we get

$$\|u_{\mathrm{ff}}(t)\|_\infty < \left(\beta M_{S_u}/\alpha\right) \|A_U\|_\infty \|U^{-1}\|_\infty e^{\alpha(t-t_0)}$$

$$:= M e^{\alpha(t-t_0)}$$

which concludes the proof. □

The Lemma states that the preactuation input tends to zero exponentially, as we go back in time from the start of the maneuver at t_0. The rate at which the preactuation becomes zero can be increased by moving the unstable poles of the modified inverse away from the imaginary axis at the expense of exact output tracking. The tradeoff between exact tracking of the desired output and reduction of the preactuation time is illustrated in the following example.

III. Example: Helicopter Hover Control

Here, we apply the output-tracking technique to the hover control of a Bell 205 helicopter, which has near nonhyperbolic unstable internal dynamics. We consider one of the cases studied in Ref. 26, wherein the aircraft dynamics was trimmed at a nominal 5-deg pitch attitude, with a midrange weight and a midposition center of gravity and operating in-ground effect at near sea level. The linearized model is given as[26,27]

$$\dot{x} = Ax + Bu \tag{20}$$

where

$$A = \begin{bmatrix} 0 & 0.03 & 0.18 & -0.01 & -0.42 & 0.08 & -9.81 & 0 \\ -0.10 & -0.39 & 0.09 & -0.10 & -0.72 & 0.68 & 0 & 0 \\ 0.01 & -0.01 & -0.19 & 0 & 0.23 & 0.04 & 0 & 0 \\ 0.02 & 0 & -0.41 & -0.05 & -0.27 & 0.27 & 0 & 9.81 \\ 0.03 & -0.02 & -0.88 & -0.04 & -0.57 & 0.14 & 0 & 0 \\ -0.01 & -0.02 & -0.06 & 0.07 & -0.32 & -0.71 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

$$B = \begin{bmatrix} 0.08 & 0.13 & 0 & 0 \\ -1.17 & 0.04 & 0 & 0.01 \\ 0 & -0.07 & 0 & 0.01 \\ -0.04 & 0 & 0.11 & 0.19 \\ -0.04 & 0 & 0.22 & 0.17 \\ 0.17 & 0 & 0.03 & -0.47 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

$$\boldsymbol{x} = \begin{bmatrix} U \\ W \\ Q \\ V \\ P \\ R \\ \theta \\ \chi \end{bmatrix} \tag{23}$$

where U is forward velocity, W vertical velocity, Q pitch rate, V lateral velocity, P roll rate, R yaw rate, θ pitch attitude, and χ roll attitude, and

$$\boldsymbol{u} = \begin{bmatrix} \delta_C \\ \delta_B \\ \delta_A \\ \delta_P \end{bmatrix} \tag{24}$$

where δ_C is collective, δ_B longitudinal cyclic, δ_A lateral cyclic, and δ_P tail rotor collective.

It is noted that the helicopter's actual dynamic behavior differs because of modeling errors such as nonlinearities and unmodeled dynamics. In output tracking control schemes that depend on the model, such modeling errors need to be corrected through feedback in the control scheme (see Fig. 1). In particular, modeling errors can be compensated by robust stabilization of the reference state trajectory (see, for example, Ref. 26). The goal is to develop inversion-based feedforward and reference state trajectories for use in the control scheme shown in Fig. 1. In the following, we apply the inversion technique to control the helicopter's forward, lateral, and vertical velocities and its yaw rate. The forward velocity and the yaw rate are to be kept at zero, and the desired profiles of lateral and vertical velocities and accelerations are as shown in Figs. 2 and 3.

A. Internal Dynamics

To find the internal dynamics, we begin with a change in the coordinates. Let ζ be defined as the outputs

$$\zeta(t) := \begin{bmatrix} U(t) \\ W(t) \\ V(t) \\ R(t) \end{bmatrix} \tag{25}$$

and let η be the remaining states

$$\eta(t) := \begin{bmatrix} Q(t) \\ \theta(t) \\ P(t) \\ \chi(t) \end{bmatrix} \tag{26}$$

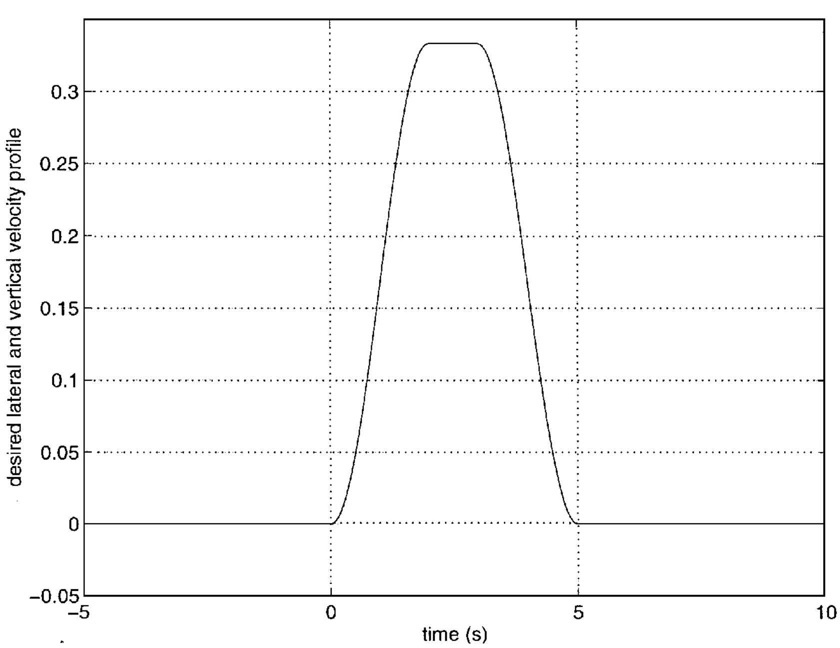

Fig. 2 Desired lateral and vertical velocity profile.

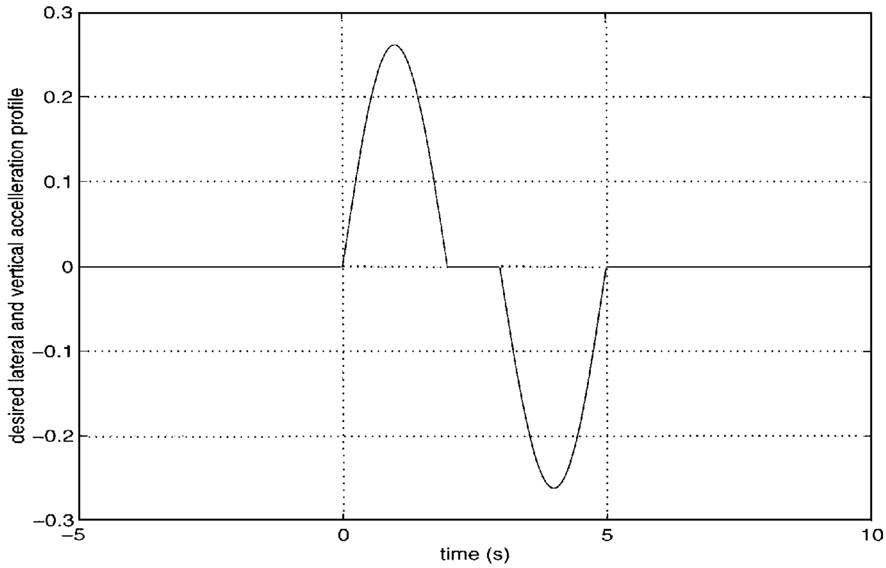

Fig. 3 Desired lateral and vertical acceleration profile.

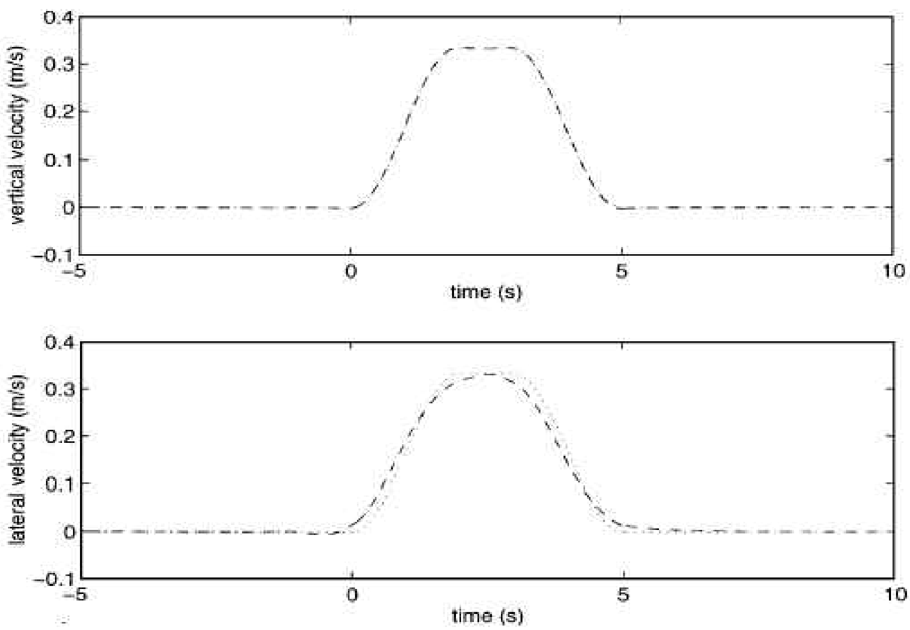

Fig. 4 Lateral and vertical velocity achieved by the inverse reference state trajectory: · · · · ·, exact-tracking case without modification of the internal dynamics and – – –, with modification.

In the (ζ, η) coordinate system, given by

$$\begin{bmatrix}\zeta(t)\\ \eta(t)\end{bmatrix} = \begin{bmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1 \end{bmatrix} \boldsymbol{x}(t)$$

$$:= T\boldsymbol{x}(t) \tag{27}$$

the system equations can be rewritten as

$$\begin{bmatrix}\dot\zeta(t)\\ \dot\eta(t)\end{bmatrix} = TAT^{-1}\begin{bmatrix}\zeta(t)\\ \eta(t)\end{bmatrix} + TB\boldsymbol{u}(t)$$

$$:= \begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\begin{bmatrix}\zeta(t)\\ \eta(t)\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}\boldsymbol{u}(t) \tag{28}$$

Given a desired output trajectory and the desired output's time derivatives, $[\zeta_d(\cdot), \dot\zeta_d(\cdot)]$, the exact output tracking control law (see Sec. II.D) is found from Eq. (28) as

$$\boldsymbol{u}_{\mathrm{ff}}(t) = B_1^{-1}[\dot\zeta_d(t) - A_1\zeta_d(t) - A_2\eta(t)] \tag{29}$$

With this control law, the inverse system becomes [from Eq. (28)]

$$\dot\zeta(t) = \dot\zeta_d(t) \tag{30}$$

$$\begin{aligned}\dot\eta(t) &= A_4\eta(t) + A_3\zeta(t) + B_2\boldsymbol{u}(t)\\ &= \left[A_4 - B_2B_1^{-1}A_2\right]\eta(t) + B_2B_1^{-1}[\dot\zeta_d(t) - A_1\zeta_d(t)]\\ &:= A_\eta\eta(t) + B_2B_1^{-1}[\dot\zeta_d(t) - A_1\zeta_d(t)]\end{aligned} \tag{31}$$

The problem is solved by finding a bounded solution to the internal dynamics (31). However, the bounded solution found through stable inversion is noncausal and could require a large preactuation time if the poles of the internal dynamics are unstable and lie close to the imaginary axis in the complex plane. For this particular example, there are two such complex conjugate poles near the imaginary axis, $0.0425 \pm 4.3055i$. We modify the exact tracking control law (29) to

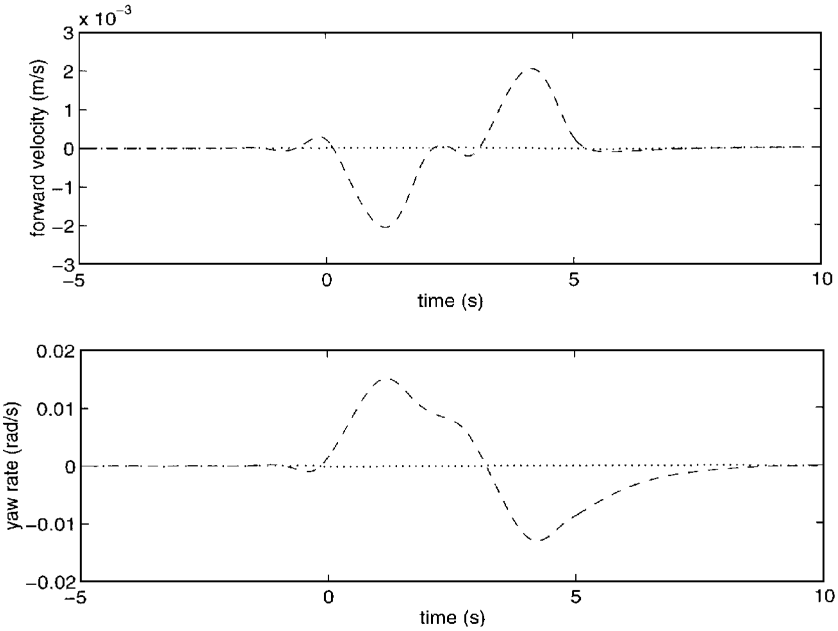

Fig. 5 Forward velocity and yaw rate achieved by the inverse reference state trajectory, the desired value is zero: · · · · ·, exact-tracking case without modification of the internal dynamics and – – –, with modification.

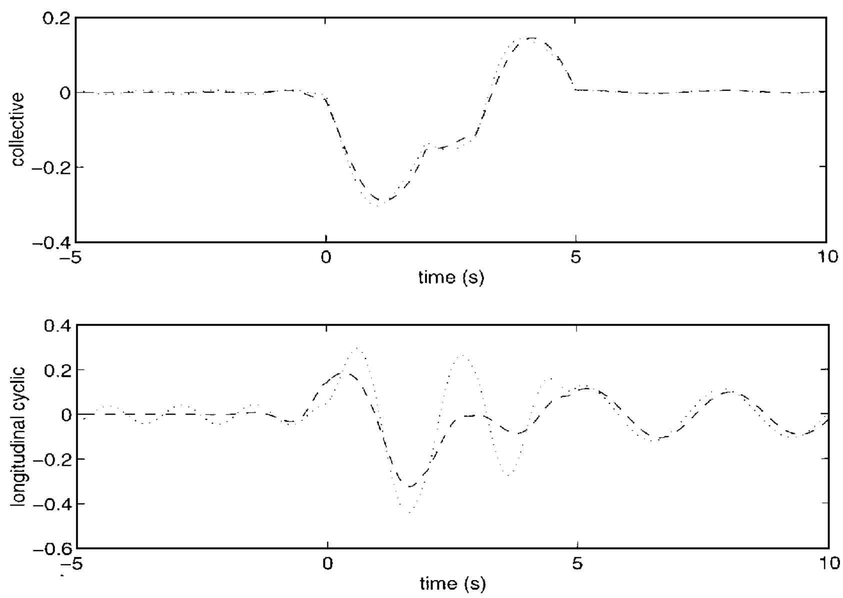

Fig. 6 Feedforward inputs: · · · · ·, without modification of internal dynamics and – – –, with modification.

shift these poles away from the imaginary axis to $2 \pm 4.3055i$. This is described next.

B. Modified Inverse System

Following the approach described in Sec. II.C, we modify the internal dynamics by adding a term $v(t)$ to the control law (29) to obtain

$$\boldsymbol{u}_{\mathrm{ff}}(t) = B_1^{-1}[\dot{\boldsymbol{\zeta}}_d(t) - A_1\boldsymbol{\zeta}(t) - A_2\boldsymbol{\eta}(t) + \boldsymbol{v}(t)] \tag{32}$$

Substituting this control law into Eqs. (30) and (31), the modified inverse system is obtained as

$$\begin{aligned}\begin{bmatrix}\dot{\boldsymbol{e}}_\zeta(t)\\ \dot{\boldsymbol{\eta}}(t)\end{bmatrix} &= \begin{bmatrix}0 & 0\\ \left(A_3 - B_2B_1^{-1}A_1\right) & A_\eta\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_\zeta(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} + \begin{bmatrix}I\\ B_2B_1^{-1}\end{bmatrix}\boldsymbol{v}(t)\\ &\quad + \begin{bmatrix}0 & 0\\ \left(A_3 - B_2B_1^{-1}A_1\right) & B_2B_1^{-1}\end{bmatrix}\begin{bmatrix}\boldsymbol{\zeta}_d(t)\\ \dot{\boldsymbol{\zeta}}_d(t)\end{bmatrix}\\ &:= \hat{S}\begin{bmatrix}\boldsymbol{e}_\zeta(t)\\ \boldsymbol{\eta}(t)\end{bmatrix} + G_v\boldsymbol{v}(t) + G_y\boldsymbol{Y}_d(t)\end{aligned} \tag{33}$$

where $\boldsymbol{e}_\zeta(t) := \boldsymbol{\zeta}(t) - \boldsymbol{\zeta}_d(t)$. The poles of the inverse system can be moved to any desired location by using the control $\boldsymbol{v}(t)$ because $(\hat{S}, G_v)$ is controllable. However, such modifications, aimed at reducing preactuation time, will also lead to a loss of precision in output tracking. This tradeoff between the reduction of preactuation time and the loss of precision in tracking is illustrated through simulation.

C. Simulation Results and Discussion

Two sets of simulations were performed. First, stable inversion was applied to the original system, which leads to exact output tracking inverse input-state trajectories. Second, simulations were performed when the unstable poles of the inverse system are moved from $0.0425 \pm 4.3055i$ to $2 \pm 4.3055i$ for reducing the amount of preactuation time required. Further, the inverse system also has four poles at the origin, corresponding to four pure integrators for $[\boldsymbol{e}_\zeta(\cdot)]$ dynamics, which were moved to -1, -2, -3, and -4 for stability of the numerical integration scheme.

Figures 2 and 3 show the desired output trajectories for the lateral and vertical motions (corresponding to unit displacements in the two directions), while the forward velocity and yaw rate were to be

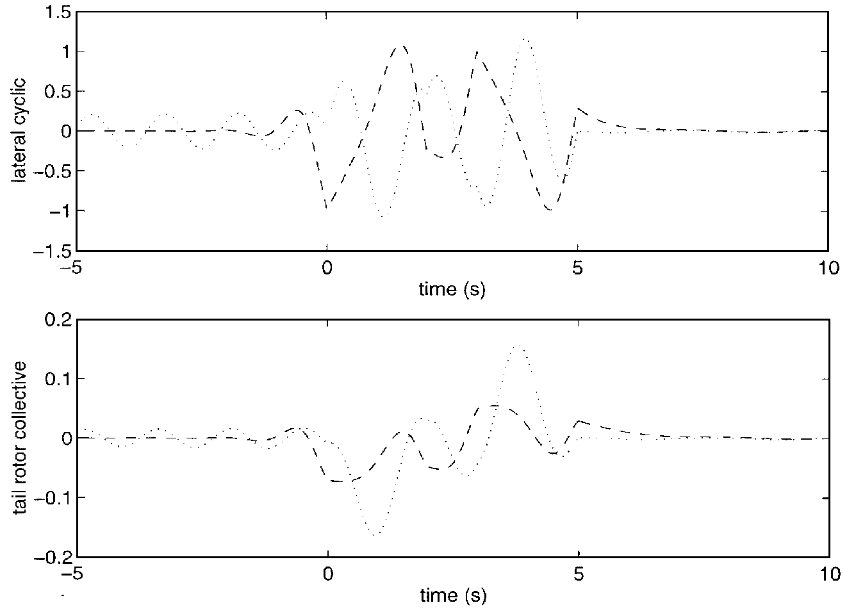

Fig. 7 Feedforward inputs: · · · · ·, without modification of internal dynamics and – – –, with modification.

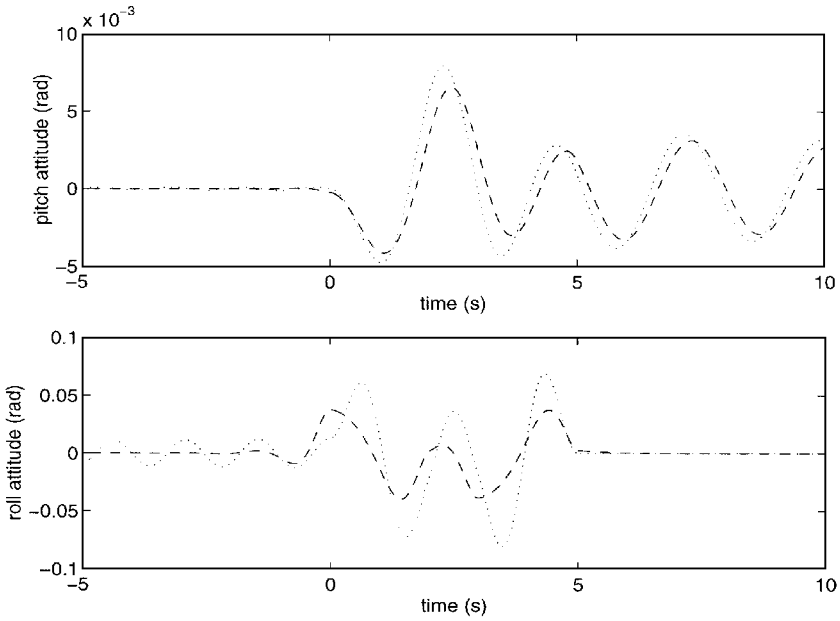

Fig. 8 Internal dynamics: · · · · ·, without modification of internal dynamics and – – –, with modification.

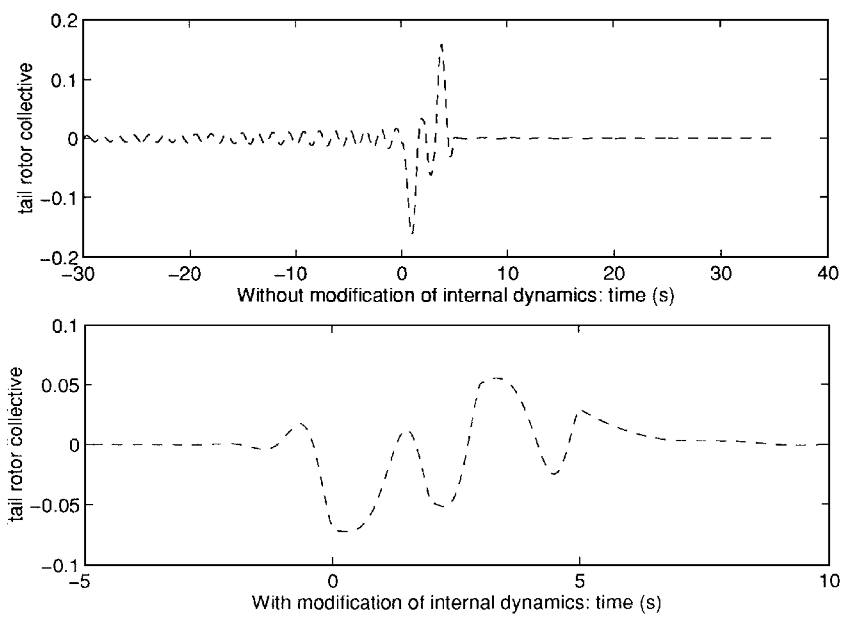

Fig. 9 Comparison of required preactuation in the feedforward.

maintained at zero; the maneuver starts at time $t = 0$. Figures 4 and 5 show the output trajectories achieved by the inverse state trajectory $x_{\text{ref}}(\cdot)$, which is to be used as a reference trajectory in the feedback scheme shown in Fig. 1. The corresponding feedforward inputs are shown in Figs. 6 and 7. Note here that the feedforward inputs are nonzero before the start of the maneuver, i.e., time $t < 0$, and hence preactuation is required.

Figures 4 and 5 show that exact output tracking reference state trajectories can be found, even when the internal dynamics is unstable, through the stable inversion approach. The stable inversion technique yields bounded solutions to the unstable internal dynamics, i.e., the pitch and roll motions are bounded, as shown in Fig. 8. However, the feedforward input found through exact inversion requires substantial preactuation time, as shown in Figs. 6 and 7, i.e., the preactuation remains nonzero for a significant time before the start of the maneuver at $t = 0$. Figure 9 shows that about 30 s of preactuation is needed to apply the inverse of the original system for output tracking; in contrast, modification of the internal dynamics reduces the preactuation needed from 30 to 1 s (see Fig. 9). As seen in Figs. 4 and 5, the output trajectories are still tracked well by the modified inverse. Further, this substantial reduction in preactuation time is achieved with similar control efforts and with similar roll and pitch motions (see Figs. 6–8). Thus, the approach presented here allows a tradeoff between precision tracking and the amount of preactuation that is acceptable. Future work will generalize the results to nonlinear nonminimum phase systems with nonhyperbolic internal dynamics.

IV. Conclusions

A technique to achieve output tracking for nonminimum phase linear systems with nonhyperbolic and near nonhyperbolic internal dynamics was presented. This approach is an integration of the stable inversion technique that aims at exact tracking with the approximation approach that modifies the internal dynamics to achieve desirable performance. The method was applied to an example helicopter hover control problem to illustrate the effects of modifying the internal dynamics. It was shown that substantial reduction in preactuation time is possible by giving up some of the precision in tracking, thus making the stable inversion approach viable for practical application.

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